LAST TIME: FUBINI'S THEOREM 

If $f$ is cts on $R = [a,b] \times [c,d]$, then $\int_{y=c}^{d} \int_{x=a}^{b} f(x,y)\,dy = \iint_R f(x,y)\,dA = \int_{x=a}^{b} \int_{y=c}^{d} f(x,y)\,dy\,dx$.

ex: compute $\iint_R \frac{x}{1+xy}\,dA$ on $R = [0,1] \times [0,1]$

sol 1: $\iint_R \frac{x}{1+xy}\,dA = \int_{x=0}^{1} \int_{y=0}^{1} \frac{x}{1+xy}\,dy\,dx$

inner integral: $\int_{y=0}^{1} \frac{x}{1+xy}\,dy$ $\left( \begin{array}{l} u(y) = 1+xy \\ du = x\,dy \end{array} \right.$

$= \int_{y=0}^{1} \frac{1}{1+xy} \cdot x\,dy = \int_{y=0}^{1} \frac{1}{u}\,du = \ln|u| \Big|_{y=0}^{1} = \ln|1+xy| \Big|_{y=0}^{1}$

$= \ln(1+x) - \ln(1+0) = \ln(1+x)$

outer integral: $\int_{x=0}^{1} \ln(1+x)\,dx = \iint_R \frac{x}{1+xy}\,dA$

$= \left[ x\ln(1+x) - \int \frac{x+1-1}{1+x}\,dx \right]_{x=0}^{1}$ $\left( \begin{array}{ll} u = \ln(1+x) & dv = dx \\ du = \frac{1}{1+x}dx & v = x \end{array} \right.$

$= \left[ x\ln(1+x) - \int \left(1 - \frac{1}{1+x}\right) dx \right]_{x=0}^{1}$

$= \left[ x\ln(1+x) - (x - \ln(1+x)) \right]_{x=0}^{1}$

$= (\ln(2) - (1 - \ln(2))) - (0 - (0 - \ln(1))) = 2\ln(2) - 1$

sol 2: inner integral: $\int_{x=0}^{1} \frac{x}{1+xy}\,dx$

- in principle: could try $\left( \begin{array}{l} u(x) = 1+xy \rightarrow x = \frac{u-1}{y} \\ du = y\,dx \end{array} \right.$

$= \int_{x=0}^{1} \frac{\left(\frac{u-1}{y}\right)}{u} \cdot \frac{1}{y}\,du$

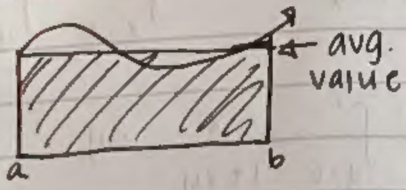
$= \boxed{\frac{1}{y^2}} \int_{x=0}^{1} \frac{u-1}{u}\,du$

distributes discontinuity at $y=0$

**Exercise:** compute $\iint_R ye^{-xy}\, dA$ on $R = [0,2] \times [0,3]$. write out both possible orders of integration.

**POINT:** Sometimes one order is more computable than another order.

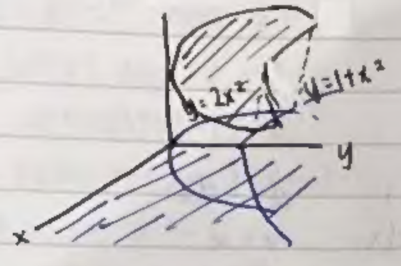
**Def$^n$:** The average value of function $f(x,y)$ on Region $R$ is
$$\frac{1}{\text{Area}(R)} \iint_R f(x,y)\, dA$$




**Goal:** Integrate over more than just rectangles...

**ex:** compute the (net) volume of the solid bounded by
$y = 2x^2$, $y = 1 + x^2$, $z = x + 2y$ and $z = 0$

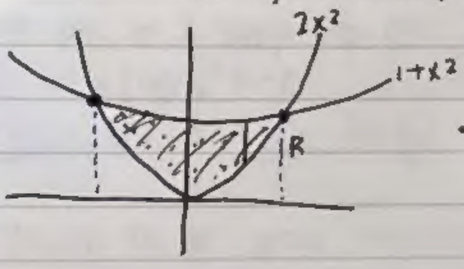
**Picture:**




**Net volume:** $\iint_R ((x+2y)-0)\, dA$ over

$R : \{ (x,y) : (x,y)$ between $y = 2x^2$ & $y = 1+x^2$

Now a picture of the region in the xy-plane:




- For fixed $x$, we know $2x^2 \le y \le 1 + x^2$
- To find $x$-bounds, solve $\left( \begin{array}{l} 2x^2 = 1+x^2 \\ \text{iff } x = \pm 1 \end{array} \right)$

$\therefore R = \{ (x,y) : -1 \le x \le 1,\ 2x^2 \le y \le 1 + x^2 \}$

Thus, because our parameterization of $R$ is nice, we can write our double integral as an iterated integral!

**Sol:** We just saw $R = \{ (x,y) : -1 \le x \le 1,\ 2x^2 \le y \le 1+x^2 \}$

$$\therefore \iint_R (x+2y)\, dA = \int_{x=-1}^{1} \int_{y=2x^2}^{1+x^2} (x+2y)\, dy\, dx$$

$$= \int_{x=-1}^{1} \left[ xy + y^2 \right]_{y=2x^2}^{1+x^2} dx = \int_{x=-1}^{1} (x(1+x^2) + (1+x^2)^2 - (x(2x^2)) + (2x^2)^2))\, dx$$

$$= \int_{x=-1}^{1} x(1 + x^2 - 2x^2) + ((1+x^2)^2 - (2x^2)^2))\, dx =$$

→

$= \int_{x=-1}^{1} (x(1-x^2) + (1+x^2+2x^2)(1+x^2-2x^2))\,dx$

$= \int_{x=-1}^{1} (1+x+3x^2)(1-x^2)\,dx$

$= \int_{x=-1}^{1} (1+x+2x^2-x^3-3x^4)\,dx$

$= \left[ x + \frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4 - \frac{3}{5}x^5 \right]_{x=-1}^{1}$

$= \left(1+\frac{1}{2}+\frac{2}{3}-\frac{1}{4}-\frac{3}{5}\right) - \left(-1+\frac{1}{2}-\frac{2}{3}-\frac{1}{4}-\frac{3}{5}\right) = 2\cdot 1 + 0 + 2\cdot\frac{2}{3} - 0 - 2\cdot\frac{3}{5} = 2\left(1+\frac{2}{5}-\frac{3}{5}\right) = 2\left(\frac{15+10-9}{15}\right)$

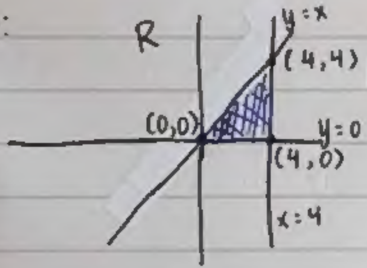
$= \frac{32}{15}$ ☒.

TAKEAWAY: If R is parametrized by something like $R = \{(x,y) : c_1 \le x \le c_2,\ g_1(x) \le y \le g_2(x)\}$, then $\iint_R f(x,y)\,dA = \int_{x=c_1}^{c_2} \int_{y=g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$

Similarly, $R = \{(x,y) : c_1 \le y \le c_2,\ g_1(y) \le x \le g_2(y)\}$ yields
$\iint_R f(x,y)\,dA = \int_{y=c_1}^{c_2} \int_{x=g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$

ex: compute $\iint_R y^2 e^{xy}\,dA$ for R bounded by $y=x$, $y=0$, $x=4$

Picture:




$R = \{(x,y) : 0 \le y \le 4,\ y \le x \le 4\}$
$R = \{(x,y) : 0 \le x \le 4,\ 0 \le y \le x\}$

sol: $\iint_R y^2 e^{xy}\,dA = \underbrace{\int_{y=0}^{4} \int_{x=y}^{4} y^2 e^{xy}\,dx\,dy}_{\text{easier to do...}} = \int_{x=0}^{4} \int_{y=0}^{4} y^2 e^{xy}\,dy\,dx$

$\iint_R y^2 e^{2xy}\,dA = \int_{y=0}^{4} \int_{x=y}^{4} y^2 e^{xy}\,dx\,dy$

inner: $\int_{x=y}^{4} y^2 e^{xy}\,dx = \int_{x=y}^{4} ye^{xy}\,\underbrace{y\,dx}_{du} = \int_{x=y}^{4} ye^{u}\,du = y[e^u]_{x=y}^{4}$

$\begin{cases} u = xy \\ du = y\,dx \end{cases}$ $= y[e^{xy}]_{x=y}^{4} = y(e^{4y} - e^{y^2}) = ye^{4y} - ye^{y^2}$

$\therefore \iint_R y^2 e^{xy} = \int_{y=0}^{4} (ye^{4y} - ye^{y^2})\,dy$

$= \int_{y=0}^{4} ye^{4y}\,dy - \int_{y=0}^{4} ye^{y^2}\,dy$   IBP: $\begin{cases} u = y & dv = e^{4y}\,dy \\ du = dy & v = \frac{1}{4}e^{4y} \end{cases}$ / sub: $\begin{cases} w = y^2 \\ dw = 2y\,dy \end{cases}$ →

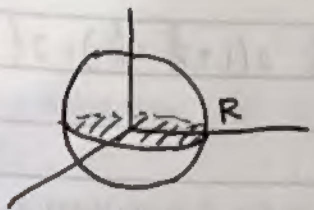
$$= \left[ \frac{1}{4} y e^{4y} - \int \frac{1}{4} e^{4y} dy - \int e^{x} dw \right]_{y=0}^{4} = \left[ \frac{1}{4} 4 e^{4y} - \frac{1}{16} e^{4y} - \frac{1}{2} e^{y^2} \right]_{y=0}^{4}$$

$$= \left( e^{16} - \frac{1}{6} e^{16} - \frac{1}{2} e^{16} \right) - \left( 0 - \frac{1}{16} - \frac{1}{2} \right) = \frac{1}{16}\left( 1 - e^{16} \right) + \frac{1}{2} - \frac{1}{2} e^{2}$$ ∎

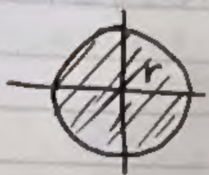
↳ <u>Motivating question</u>: what is the volume of the sphere?

<u>Set up</u>: $S : \{ (x,y,z) : x^2 + y^2 + z^2 = r^2 \}$

$z = \pm \sqrt{r^2 - x^2 - y^2}$




• need a nice parameterization of R




$R : \{ (x,y) : -r \leq x \leq r, \ -\sqrt{r^2 - x^2} \leq y \leq \sqrt{r^2 - x^2} \}$